Lecture 20: Inverse matrices overview and orthogonality

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18.5 Invertible Matrix Theorem

Let A be an *nxn* matrix. Then, the following statements are equivalent:

1) **A is invertible**
2) $rank(A) = n$ (there are n leading ones; for every column in the REF there's a leading 1)
3) $Ax = 0$ has only the trivial solution ($x = 0$)
4) $Ax = b$ is consistent for every $b \in \mathbb{R}^n$
5) $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$
6) The RREF of A is I (the identity matrix)
7) Null(A)= $\{x \in \mathbb{R}^n \mid Ax = 0\} = \{0\}$
8) Col(A)= $\{Ax \mid x \in \mathbb{R}^n\} = \mathbb{R}^n$
9) Row(A)=$\mathbb{R}^n$
10) rank$(A^T) = n$

    proof:

    rank(A)

    $= \dim(Row(A))$

    $= \dim(Col(A))$

    $= \dim(Row(A^T))$

    $= rank(A^T)$
11) The columns of A are linearly independent.
12) The rows of A are linearly independent.
13) The columns of A span $\mathbb{R}^n$
14) The rows of A span $\mathbb{R}^n$
15) The columns of A are a basis of $\mathbb{R}^n$
16) The rows of A are a basis of $\mathbb{R}^n$
17) A^T is invertible.

understand what's behind these equivalences

→ F15-T4 6b

19 Orthogonality

Recall

$u, v \in \mathbb{R}^n$

$u \perp v$ (u and v are ortogonal)

$<=> u \cdot v = 0$

Properties of the dot product:

i) $u \cdot u \geq 0$ and $u \cdot u = 0 <=> u = 0$

ii) $u \cdot v = v \cdot u \rightarrow$ "symmetric"

iii) $\left.\begin{array}{l}(au + bv) \cdot w = au \cdot w + bv \cdot w \\ u \cdot (av + bw) = au \cdot v + bu \cdot w\end{array}\right\}$ "bilinear"

Example

$$\begin{pmatrix}1\\-1\\0\\1\end{pmatrix} \perp \begin{pmatrix}1\\1\\1\\0\end{pmatrix}?$$

$$\begin{pmatrix}1\\-1\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\\0\end{pmatrix} = 1 \cdot 1 + (-1) \cdot 1 + 0 \cdot 1 + 1 \cdot 0 = 0$$

they are $\perp$

Observation

$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

are pairwise orthogonal and a basis of $\mathbb{R}^3$

Let $v \in \mathbb{R}^3$

$v = a \cdot e_1 + b \cdot e_2 + c \cdot e_3 \quad (*)$

If we want to know what a is:

Take $(*)$ and take the dot product of with e_1 on both sides:

$$e_1 \cdot v = e_1 \cdot (ae_1 + be_2 + ce_3)$$

$$<=> e_1 \cdot v = a\underbrace{e_1 \cdot e_1}_{1} + b\underbrace{e_1 \cdot e_2}_{0} + c\underbrace{e_1 \cdot e_3}_{0}$$

$$<=> e_1 \cdot v = a$$

We only have to compute $e_1 \cdot v$ to get a, similarly:

$b = e_2 \cdot v$

$c = e_3 \cdot v$

19,2 Orthogonal sets of vectors

Definition

A set of vectors $\{v_1, \dots, v_m\} \subseteq \mathbb{R}^n$ is called orthogonal if:

(1) $v_i \cdot v_j = 0$ for $1 \le i \le j \le m$

(2) $v_i \neq 0$ for $1 \le i \le m$

Example

$e_1 = \begin{pmatrix}2\\-1\\0\end{pmatrix}, e_2 = \begin{pmatrix}1\\2\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\5\end{pmatrix}$

$e_1 \cdot e_2 = 0$ ✓ $\quad e_1 \neq 0$ ✓

$e_1 \cdot e_3 = 0$ ✓ $\quad e_2 \neq 0$ ✓ yes! orthogonal

$e_2 \cdot e_3 = 0$ ✓ $\quad e_3 \neq 0$ ✓

Moreover, the set is called **orthonormal** if, in addition to (1) and (2):

(3) $v_1 \cdot v_1 = ||v_1||^2 = 1$ for all $1 \le i \le m$

NOTE:

If $\{v_1, \dots, v_m\}$ is orthogonal, then

$\frac{v_1}{||v_1||}, \dots, \frac{v_m}{||v_m||}$ is orthonormal

length 1

Theorem

Orthogonal sets are linearly independent.